Day 6

For circle rotations by angle θ,

$$R_\theta : \mathbb{R}/_{2\pi\mathbb{Z}} \longrightarrow \mathbb{R}/_{2\pi\mathbb{Z}}$$
$$[x] \longmapsto [x+\theta]$$

The conditions for an orbit to be periodic is that $\theta = \frac{p}{q} 2\pi$ where p, q are integers.

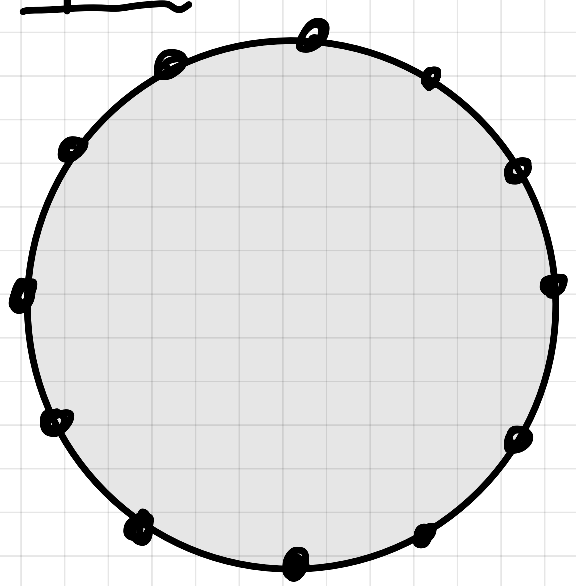
Similarly for $R_\alpha : \mathbb{R}/_{\mathbb{Z}} \to \mathbb{R}/_{\mathbb{Z}}$
$$[x]_{\mathbb{Z}} \mapsto [x+\alpha]_{\mathbb{Z}}$$

the orbits are periodic if $\alpha = \frac{p}{q}$ for some $p \in \mathbb{Z}$, $q \in \mathbb{N}$. (i.e. α is a rational number)

i.e. $\alpha \in \mathbb{Q}$ = the set of rational numbers.

Exercise. Find a formula for $\mathbb{Q}$ using set builder notation

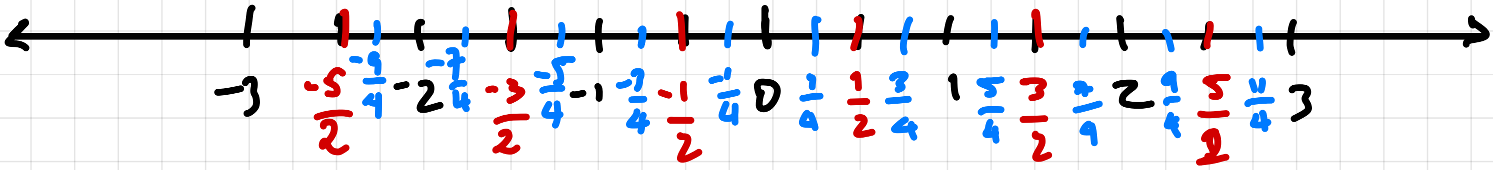
Sparse:



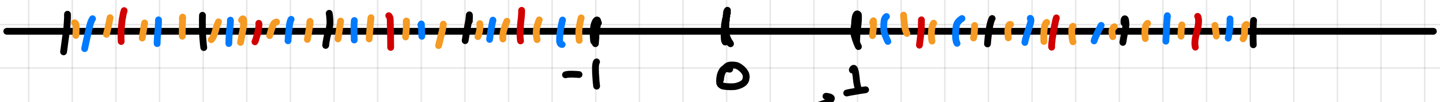
Opposite of finite/sparse?

On $\mathbb{R}$, what does it mean for a set to be dense?

Example $\mathbb{Q}$ is dense in $\mathbb{R}$.



Example $\{x \in \mathbb{Q} : |x| \geq 1\}$ is <u>not</u> dense in $\mathbb{R}$



$(-1, 1) \longrightarrow$ gap in $\{x \in \mathbb{Q} : |x| \geq 1\}$

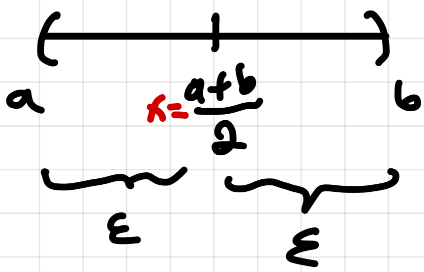
then this set is <u>not</u> dense in $\mathbb{R}$

Definition $A \subset \mathbb{R}$ is <u>not dense</u> if there exists an interval (a,b) that $(a,b) \cap A = \phi$.

Definition $A \subset \mathbb{R}$ is <u>dense</u> if for any interval (a,b), $(a,b) \cap \mathbb{R} \neq \phi$.

Theorem $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof we pick an interval, $(a,b) \subset \mathbb{R}$



$x = \frac{a+b}{2}$ center

and $\varepsilon = x - a = \frac{b-a}{2} > 0$

write x in its decimal expansion:

$$x = x_n x_{n-1} \dots x_0 . x_{-1} x_{-2} x_{-3} \dots$$

$$(\text{e.g } 947.32017\dots)$$

Define $x_N = x_n x_{n-1} \dots x_0 . x_{-1} x_{-2} \dots x_{-N}$ a truncation of the decimals in x at the Nth decimal.

Observe that

$$x - x_N = 0.\underbrace{00\cdots0}_{N\ 0's}x_{-N-1}x_{-N-2}x_{-N-3}\cdots < 0.\underbrace{0000\cdots0}_{N-1\ 0's}1 = \frac{1}{10^{N-1}}$$

ex. $0.000356 < 0.001$

no matter how small ε is we can find an N such that $\frac{1}{10^{N-1}} = 10^{-(N-1)} < \varepsilon$ (Since $10^{-(N-1)} \to 0$ as $N \to \infty$)

thus $|x - x_N| < \varepsilon$

$\boxed{\text{Exercise} \quad x_N \in (a,b)}$

Since x_N is a rational, i.e $x_N \in \mathbb{Q}$ and the exercise above, then $x_N \in \mathbb{Q} \cap (a,b)$. proving that $\mathbb{Q}$ is dense in $\mathbb{R}$. $\blacksquare$

Tomorrow: we will prove that there are only two types of subgroups in $\mathbb{R}$.

Either: $H = c\mathbb{Z}$ for some c.

or H is a dense subgroup.